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Deautonomizing integrable non-QRT mappings

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Abstract
We examine a class of previously derived integrable mappings which do not belong to the QRT family and show that they can be extended to non-autonomous forms without loss of the integrable character. We derive more non-QRT integrable mappings, obtain their non-autonomous forms and show how they can be integrated.

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1. Introduction

Integrable second-order mappings constitute an excellent example of sustained progress. While a quarter of a century ago just a handful of such integrable systems could be exhibited, we are today in a situation of abundance, thanks to the concentrated efforts of several teams all over the world. Foremost among these integrable second-order systems are the mappings belonging to the QRT family [1]. Starting from specific integrable mappings obtained from reductions of integrable differential-difference systems, Quispel, Roberts and Thomson (QRT) were led to the proposal of a five-parameter family of second-order mappings of the form

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}, \tag{1.1}$$

where the f_i are specific polynomials of order not higher than 4. This, so-called symmetric, QRT mapping possesses an invariant of the form:

$$(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n+1}x_n(x_{n+1} + x_n) + (\gamma_0 + K\gamma_1)\left(x_{n+1}^2 + x_n^2\right) \\ + (\epsilon_0 + K\epsilon_1)x_{n+1}x_n + (\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\kappa_0 + K\kappa_1) = 0, \tag{1.2}$$

where K plays the role of the integration constant. Moreover, it was shown that the solution of the mapping can be expressed in terms of elliptic functions, of which it is a sampling over a discrete, equidistant set of points. A generalization of the mapping (1.1) to an eight-parameter one was proposed by QRT under the name 'asymmetric'. It is a system of two first-order mappings and possesses an invariant which is a ratio of two biquadratic polynomials. Its integration in terms of elliptic functions was given in [2, 3].

An interesting question is whether there exist integrable second-order mappings not belonging to the QRT parametrization. As a matter of fact, several such examples do exist. A family of such mappings is the one discovered by Hirota, Kimura and Yahagi (HKY) [4] who, while investigating third-order mappings obtained systems which could be integrated to second-order mappings with *biquartic* invariants. Several more such mappings were discovered in [5]. They were obtained through the procedure of appropriate *autonomization* of the discrete Painlevé equations. As an illustration, we start from the q-$\mathrm{P_V}$ which was introduced in [6]:

$$y_n y_{n-1} = \frac{(x_n - aq^n)(x_n - bq^n)}{1 - px_n} \tag{1.3a}$$

$$x_{n+1} x_n = \frac{(y_n - cq^n)(y_n - dq^n)}{1 - ry_n} \tag{1.3b}$$

with the constraint $cd = qab$. We consider an autonomous reduction with $q = -1$, which imposes $a + b = c + d = 0$, and $c^2 = -a^2$. Moreover, we take $p = 1$, $r = i$ and rescale y as $y \to -iy$. We find

$$y_n y_{n-1} = \frac{x_n^2 - a^2}{x_n - 1} \tag{1.4a}$$

$$x_{n+1} x_n = \frac{y_n^2 - a^2}{y_n - 1}. \tag{1.4b}$$

From (1.4), we can obtain the symmetric reduction, identifying $y_{n-1} = X_{2n-1}$, $x_n = X_{2n}$, $y_n = X_{2n+1}$, etc and demanding that (1.4*b*) be just the upshift of (1.4*a*). Denoting, for simplicity, the new variable by x rather than X leads to the symmetric, one-component, form of this mapping:

$$x_{n+1} x_{n-1} = \frac{x_n^2 - a^2}{x_n - 1}. \tag{1.5}$$

The invariant for (1.5) is simply

$$K = \frac{x_n^2 x_{n+1}^2 (x_n - x_{n+1})^2 - 2x_n x_{n+1}(x_n + x_{n+1})((x_n - x_{n+1})^2 - a^2) + \left(x_n^2 + x_{n+1}^2 - a^2\right)^2}{x_n^2 x_{n+1}^2}. \tag{1.6}$$

More examples of HKY-type mappings were discovered in [7, 8].

At this point, one may wonder whether the integration of the mappings of the HKY family is different from that of QRT mappings. In [9], we have shown that this is not so. The integration of HKY-type mappings follows the general procedure for the integration of integrable mappings in the plane. One starts from the invariant curve, i.e. an invariant such as (1.2) where we put $x_n \to x$ and $x_{n+1} \to y$. Since the mapping is generically an automorphism of infinite order, one expects the invariant curve to be of genus 0 or 1. (The genus can be computed algorithmically following the procedure proposed by van Hoeij [10].) If the genus is 1, the curve is birationally equivalent to a curve of the form

$$v^2 - 4u^3 + \alpha u + \beta = 0. \tag{1.7}$$

The precise method for the construction of the canonical form follows again the method proposed by van Hoeij. One chooses a point x, y which in turns fixes the value of K.

Then one algorithmically constructs $u(x, y)$ and $v(x, y)$ which satisfy the canonical relation $v^2 - 4u^3 + \alpha u + \beta = 0$. This last relation is parametrized in terms of the Weierstraß elliptic function $u = \wp$ and its derivative $v = \wp'$. The action of the initial mapping interpreted at the level of the canonical form (1.7) is just a shift from $\wp(z)$ to $\wp(z + \delta)$ (where the step δ is not curve-independent). Once the solution of (1.7) in terms of elliptic functions is given, one can construct the parametrization of the initial curve using the inverse transformation $x = x(u, v)$, $y = y(u, v)$. It is obvious that this construction may be particularly cumbersome and is not global in the sense that it must be carried out for fixed value of the invariant K. Still, in some deep sense, the integrability of the HKY-type mappings is not fundamentally different of that of the mappings of the QRT family.

One of the most interesting aspects of the QRT mappings is that they can be extended to non-autonomous forms, i.e. systems where the independent variable appears explicitly in the coefficients. This deautonomization of the QRT mappings has led to the discovery of the discrete analogues of the Painlevé equations and spurred a whole new direction of research. It is thus natural to ask whether non-QRT mappings are also amenable to deautonomization. As far as the mappings obtained in [5] by 'autonomizing' discrete Painlevé equations are concerned, the answer is obviously trivial. However, since more examples of integrable non-QRT mappings than those of [5] are known, the possibility of nontrivial integrable deautonomization does exist. In what follows we shall address this precise question. In section 2, we shall show how one can deautonomize some HKY-type mappings obtained in [8]. Section 3 will be devoted to the construction of a new class of non-QRT mappings for which we shall produce non-autonomous forms before proceeding to their integration.

2. Integrable mappings of non-QRT type

In [8] we examined, from the point of view of integrability, a family of mappings which were by construction of non-QRT type. The general expression of those mappings is

$$x_{n+1} = x_{n-1} \frac{P(x_n)}{Q(x_n)}, \tag{2.1}$$

where P and Q are polynomials. Our approach was to postulate a form for P, Q and single out the integrable cases through the application of integrability criteria like singularity confinement [11] and algebraic entropy [12]. Two cases were separately investigated, that of linear P and Q and that of quadratic ones. In the linear case, we found that the following mapping

$$x_{n+1} = x_{n-1} \frac{b}{a} \frac{x_n - b}{x_n - a} \tag{2.2}$$

had zero algebraic entropy with linear degree growth of the iterates. This, as explained in [13], is an indication that the mapping is linearizable. Its explicit linearization was indeed presented in [8].

A second linearizable mapping was also identified in [8]:

$$x_{n+1} = x_{n-1} \frac{a^2}{b^2} \frac{x_n - b}{x_n - a}. \tag{2.3}$$

However, this is not really different from (2.1). In fact it suffices to invert x, a and b and then exchange a and b in order to obtain (2.3) starting from (2.2).

The case of quadratic P and Q gave more interesting results. In fact, starting from the mapping

$$x_{n+1} = x_{n-1}\, a \frac{(x_n - b)(x_n - c)}{(x_n - d)(x_n - f)}, \tag{2.4}$$

we were able to identify three integrable cases with the following canonical forms:

$$x_{n+1} = -x_{n-1}\,\frac{(x_n-\alpha)(x_n-1/\alpha)}{(x_n+\alpha)(x_n+1/\alpha)} \tag{2.5}$$

$$x_{n+1} = \mathrm{i}\; x_{n-1}\,\frac{(x_n+\mathrm{i}\alpha)(x_n+\mathrm{i}/\alpha)}{(x_n+\alpha)(x_n+1/\alpha)} \tag{2.6}$$

$$x_{n+1} = \sqrt{i}\; x_{n-1}\,\frac{x_n^2-\mathrm{i}}{x_n^2-1}. \tag{2.7}$$

Their integrability was established by the explicit construction of the their invariants. In all three cases, the invariants were ratios of polynomials of degree higher than 2, confirming the HKY character of these mappings.

Before proceeding to the deautonomization of these mappings it is interesting to show that the last two mappings are not independent. In fact (2.7) is related to a special case of (2.6). It suffices indeed to introduce the simple folding transformation $X = x^2$ and take the square of (2.7). We find

$$X_{n+1} = \mathrm{i}\; X_{n-1}\,\frac{(X_n-\mathrm{i})^2}{(X_n-1)^2} \tag{2.8}$$

which is precisely (2.6) for $\alpha = -1$.

3. Deautonomizing the non-QRT mappings

In [8] we have examined only autonomous mappings and obtained the integrable subcases. In this section we are going to show that these mappings can be extended to non-autonomous forms which are still integrable. We start with the linearizable mapping (2.2) which we write in a slightly more general form as

$$x_{n+1} = x_{n-1}\; c\,\frac{x_n-b}{x_n-a}. \tag{3.1}$$

Next we allow the quantities a, b and c to be functions of the independent variable n. We recall here that since the mapping is linearizable, the singularity confinement criterion is inapplicable. Hence, we apply the algebraic entropy integrability criterion. (We are not going to enter into the technical details here; they are well documented in the literature.) We require that the degree growth of the iterates of (3.1) be linear, so as to preserve the linearizable character. We find that the only necessary constraint is $a(n-1)c(n) = b(n+1)$ whereupon the degrees of the iterates become 1, 2, 2, 3, 3, 4, 4, 5, 5, ... We can now scale x so as to put a to 1. The constraint becomes now $b(n) = c(n-1)$ and the mapping assumes the form

$$x_{n+1} = x_{n-1}\; c_n\,\frac{x_n-c_{n-1}}{x_n-1}, \tag{3.2}$$

where c is a free function of n. The linearization of (3.2) is straightforward. We first introduce an auxiliary variable $y_n = (x_{n-1}-1)(x_n-c_{n-1})$. Expanding (3.2) and identifying terms, we find that y satisfies the linear equation $y_{n+1} - c_n y_n + c_n(c_{n-1}-1) = 0$. Thus, (3.2) is equivalent to the system

$$y_{n+1} = c_n(y_n - c_{n-1} + 1) \tag{3.3a}$$

$$x_n = c_{n-1} + \frac{y_n}{x_{n-1}-1}, \tag{3.3b}$$

which is a special case of the Gambier mapping, introduced in [14]. Next we turn to the deautonomization of (2.5) and (2.6). We start from the form (2.4) which contains the full freedom and assume that all the parameters are functions of the independent variable n. First we perform a gauge transformation on x so as to have $a = 1$. Thus, we shall work with the mapping

$$x_{n+1} = x_{n-1}\frac{(x_n - b)(x_n - c)}{(x_n - d)(x_n - f)}. \tag{3.4}$$

In order to investigate the integrability of its non-autonomous form, we shall use the singularity confinement criterion. Its application is straightforward and leads to the following constraints on the coefficients

$$d_n = b_{n+2} \tag{3.5a}$$

$$f_n = c_{n+2} \tag{3.5b}$$

$$b_n b_{n+1} c_{n+1} = b_{n+4} b_{n+3} c_{n+3} \tag{3.5c}$$

$$c_n b_{n+1} c_{n+1} = c_{n+4} b_{n+3} c_{n+3}. \tag{3.5d}$$

From the ratio of (3.5c) and (3.5d), we find that $b_n/c_n = b_{n+4}/c_{n+4}$ i.e. b/c is periodic with period 4. We can now solve system (3.5) completely:

$$b_n = pq^{(\alpha n^2+\beta n+\gamma)(-1)^n+\kappa \mathrm{i}^n+\theta(-\mathrm{i})^n+\lambda(-1)^n} \tag{3.6a}$$

$$c_n = rq^{(\alpha n^2+\beta n+\gamma)(-1)^n-\kappa \mathrm{i}^n-\theta(-\mathrm{i})^n-\lambda(-1)^n} \tag{3.6b}$$

$$d_n = pq^{(\alpha n^2+(4\alpha+\beta)n+4\alpha+2\beta+\gamma)(-1)^n-\kappa \mathrm{i}^n-\theta(-\mathrm{i})^n+\lambda(-1)^n} \tag{3.6c}$$

$$f_n = rq^{(\alpha n^2+(4\alpha+\beta)n+4\alpha+2\beta+\gamma)(-1)^n+\kappa \mathrm{i}^n+\theta(-\mathrm{i})^n-\lambda(-1)^n}. \tag{3.6d}$$

We perform a new gauge in order to absorb the factor $q^{(\alpha(n^2+2n+2)+\beta(n+1)+\alpha+\gamma)(-1)^n}$ into x. The equation now becomes

$$x_{n+1} = x_{n-1}A\frac{(x_n - B)(x_n - C)}{(x_n - D)(x_n - F)}, \tag{3.7}$$

where we have

$$A_n = q^{(4\alpha(n+1)+2\beta)(-1)^n} \tag{3.8a}$$

$$B_n = pq^{(-2\alpha(n+1)-\beta)(-1)^n+\kappa \mathrm{i}^n+\theta(-\mathrm{i})^n+\lambda(-1)^n} \tag{3.8b}$$

$$C_n = rq^{(-2\alpha(n+1)-\beta)(-1)^n-\kappa \mathrm{i}^n-\theta(-\mathrm{i})^n-\lambda(-1)^n} \tag{3.8c}$$

$$D_n = pq^{(2\alpha(n+1)+\beta)(-1)^n-\kappa \mathrm{i}^n-\theta(-\mathrm{i})^n+\lambda(-1)^n} \tag{3.8d}$$

$$F_n = rq^{(2\alpha(n+1)+\beta)(-1)^n+\kappa \mathrm{i}^n+\theta(-\mathrm{i})^n-\lambda(-1)^n}. \tag{3.8e}$$

Next we introduce new variables as follows:

$$x_{4k-1} = \frac{pr}{y_{4k-1}} \tag{3.9a}$$

$$x_{4k} = y_{4k} \tag{3.9b}$$

$$x_{4k+1} = y_{4k+1} \tag{3.9c}$$

$$x_{4k+2} = \frac{pr}{y_{4k+2}}. \tag{3.9d}$$

We can now rewrite equation (3.7) in terms of the new variable. We find the general form

$$y_{n+1}y_{n-1} = \frac{(y_n - \phi_n)(y_n - \chi_n)}{(1 - y_n/\psi_n)(1 - y_n/\omega_n)}. \tag{3.10}$$

So, with the change of variables (3.9), equation (3.4) has the same functional form as the discrete Painlevé III and in fact given the structure of its parameters, it is precisely the equation obtained by Jimbo and Sakai [15] as the discrete analogue of Painlevé VI. In order to show this explicitly, we proceed to compute the expression of ϕ, χ, ψ, ω for $n = 4k-1, 4k, 4k+1, 4k+2$. We will not go into these computational details but give directly the result. We find for even n

$$\phi_n = (pms)z_n \tag{3.11a}$$

$$\chi_n = (r/ms)z_n \tag{3.11b}$$

$$\psi_n = (sp/m)z_n^{-1} \tag{3.11c}$$

$$\omega_n = (mr/s)z_n^{-1} \tag{3.11d}$$

while for odd n we have

$$\phi_n = (pt/s)z_n \tag{3.12a}$$

$$\chi_n = (rs/t)z_n \tag{3.12b}$$

$$\psi_n = (p/st)z_n^{-1} \tag{3.12c}$$

$$\omega_n = (rst)z_n^{-1}, \tag{3.12d}$$

where $z_n = q^{-2\alpha(n+1)-\beta}$, $m = q^{\kappa+\theta}$, $s = q^{\lambda}$ and $t = q^{\mathrm{i}(\kappa-\theta)}$. Thus, (3.10) is exactly the discrete q-$\mathrm{P_{VI}}$ as we claimed, albeit in a slightly unusual form. In order to bring it in the more familiar form, we introduce a final gauge and separate explicitly the variables corresponding to even and odd indices. We have $Y_n = y_{2n}z_{2n}/\sqrt{pr}$, $X_n = y_{2n+1}z_{2n+1}/\sqrt{pr}$ which leads to the equation

$$X_nX_{n-1} = \frac{(Y_n - \nu Z_n)(Y_n - Z_n/\nu)}{(1 - \sigma Y_n)(1 - Y_n/\sigma)} \tag{3.13a}$$

$$Y_{n+1}Y_n = \frac{(X_n - \mu\tilde{Z}_n)(X_n - \tilde{Z}_n/\mu)}{(1 - \rho X_n)(1 - X_n/\rho)}, \tag{3.13b}$$

where we have put $\mu = t\sqrt{p/r}/s$, $\nu = ms\sqrt{p/r}$, $\rho = \sqrt{p/r}/(st)$ and $\sigma = s\sqrt{p/r}/m$. The independent variable enters through $Z_n = z_{2n}^2$ and we have $\tilde{Z}_n^2 = Z_nZ_{n+1}$. Equation (3.13) is indeed the canonical form [16] of q-Painlevé VI.

4. New integrable non-QRT mappings

In the previous section, we dealt with mappings already analysed in [8]. However, they are not the only ones of their kind. More integrable non-QRT mappings can be found. In this section we will analyse two new types of mappings. Their forms are inspired by the canonical forms of the QRT mapping twisted in the logic of [8]. We start with

$$\frac{x_{n+1} + x_n}{x_{n-1} + x_n} = f\,\frac{x_n^2 + ax_n + b}{x_n^2 + cx_n + d}. \tag{4.1}$$

The investigation of the integrability of (4.1) is carried out using the algebraic entropy criterion, since we expect some integrable subcases to be linearizable. We will not present here the details of this analysis but just the end result. We find that the only integrable case corresponds to $f = 1$, $c = -a$ and $d = b$. Its degree growth is 1, 2, 3, 4, 5, ... and thus we expect the mapping to be linearizable. Indeed by considering the Gambier mapping

$$y_{n+1} = y_n + a \tag{4.2a}$$

$$x_n = \frac{b + y_n x_{n-1}}{a - y_n + x_{n-1}} \tag{4.2b}$$

and eliminating y, we recover the linearizable form of (4.1)

$$\frac{x_{n+1} + x_n}{x_{n-1} + x_n} = \frac{x_n^2 + a x_n + b}{x_n^2 - a x_n + b}. \tag{4.3}$$

The mapping (4.3) possesses a transcendental conserved quantity. Indeed, from the solution of (4.2a), we have that $y_n = na + y_0$ and thus $\tan(\pi y_n/a) = \text{const}$. Solving (4.2$b$) for y, we find thus

$$\tan\left(\frac{\pi}{a}\,\frac{x_n x_{n-1} + a x_n - b}{x_n + x_{n-1}}\right) = K. \tag{4.4}$$

As a consequence of the linearizability, some of the parameters of (4.1) may be functions of the independent variable. We are thus led to examine (4.1) afresh, keeping $f = 1$, but allowing for some less stringent constraint on a, b, c, d. We require that the degree growth be the same as in the autonomous case. We find now that the constraints on the parameters are $d_n = b_{n-1}$ and $c_n = -a_{n-1}$. In order to linearize the mapping, we consider now the Gambier mapping

$$y_{n+1} = y_n \tag{4.5a}$$

$$x_n = \frac{b_{n-1} + (y_n - g_n) x_{n-1}}{g_{n-1} - y_n + x_{n-1}} \tag{4.5b}$$

and eliminating y we find

$$\frac{x_{n+1} + x_n}{x_{n-1} + x_n} = \frac{x_n^2 + (g_n - g_{n+1}) x_n + b_n}{x_n^2 + (g_n - g_{n-1}) x_n + b_{n-1}}, \tag{4.6}$$

where we have introduced the auxiliary variable g through $a_n = g_n - g_{n+1}$.

The case where the polynomials in the numerator and denominator of the rhs of (4.1) are linear is also interesting. We start from

$$\frac{x_{n+1} + x_n}{x_{n-1} + x_n} = c\,\frac{x_n + a}{x_n + b}. \tag{4.7}$$

The application of the algebraic entropy integrability criterion leads to c free while $b = -a$, and the degree growth is the same a for (4.1). The extension to a nonautonomous case is straightforward: a and c are free functions of the independent variable n. Thus, the linearizable form of the mapping is

$$\frac{x_{n+1} + x_n}{x_{n-1} + x_n} = c\,\frac{x_n + a_n}{x_n - a_{n-1}}. \tag{4.8}$$

The linearization of (4.8) is given by the Gambier mapping

$$y_{n+1} = y_n + g_{n+1} \tag{4.9a}$$

$$x_n = \frac{g_n a_{n-1} + y_n x_{n-1}}{g_n - y_n}. \tag{4.9b}$$

Elimination of y leads to (4.8) with $c_n = -g_{n+1}/g_n$. It is interesting to point out here that even in the autonomous case of constant c, the corresponding Gambier mapping is explicitly nonautonomous since in that case we have $g_n = g_0(-c)^n$. We should also remark that the linearizable case (4.8) can be obtained from (4.6) by taking $x \to 0$ and an appropriate redefinition of the auxiliary variables.

Next we analyze the mapping

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = f\,\frac{x_n^2 + a x_n + b}{x_n^2 + c x_n + d}. \tag{4.10}$$

Again we start with the purely autonomous case. We find that one linearizable case exists of the form

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \lambda^2\,\frac{x_n^2 + a x_n + 1/\lambda}{x_n^2 + a\lambda x_n + \lambda}. \tag{4.11}$$

Its linearization is given by the Gambier mapping

$$y_{n+1} = y_n/\lambda \tag{4.12a}$$

$$x_n = \lambda\frac{x_{n-1} + y_n + a}{\lambda y_n x_{n-1} - 1}. \tag{4.12b}$$

At this point, it is interesting to exhibit a case where (4.11) possesses a conserved quantity. If we take λ as a root of unity, say $\lambda^p = 1$, then from (4.12a) we have $y_{n+1}^p = y_n^p$. Solving (4.12b) for y we have

$$\left(\frac{x_{n-1} + a + a x_n/\lambda}{x_n x_{n-1} - 1}\right)^p = K. \tag{4.13}$$

Since p may be any integer, we have here an invariant of arbitrary degree.

In order to proceed to the deautonomization, it is preferable to start with the full freedom of (4.10). We find again that the mapping is integrable in one linearizable case which has the form

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \frac{b_{n+1}x_n^2 + a_n x_n + b_n}{b_{n-1}x_n^2 + a_{n-1}x_n + b_n}. \tag{4.14}$$

Its linearization is given by the Gambier mapping

$$y_{n+1} = y_n \tag{4.15a}$$

$$x_n = \frac{b_n x_{n-1} + y_n + a_{n-1}}{y_n x_{n-1} - b_{n-1}}. \tag{4.15b}$$

Next we turn to the case where the right-hand side of (4.10) is not a ratio of quadratic but rather of linear polynomials. Two cases can be distinguished here. The first corresponds to a degenerate case of (4.14) where the numerator and denominator have one common factor. This happens whenever a and b satisfy the constraint

$$(a_n - a_{n-1})(a_{n-1}b_{n+1} - a_n b_{n-1}) - b_n(b_{n+1} - b_{n-1})^2 = 0 \tag{4.16}$$

in which case (4.14) degenerates to

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \frac{b_{n+1}(a_n - a_{n-1})x_n + b_n(b_{n+1} - b_{n-1})}{b_{n-1}(a_n - a_{n-1})x_n + b_n(b_{n+1} - b_{n-1})}. \tag{4.17}$$

The autonomous limit of (4.17) can be easily obtained. We find that in this case, the constraint is just $a = \pm(1+\lambda)$ and the mapping becomes

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \frac{1 \pm x_n\lambda}{1 \pm x_n/\lambda}. \tag{4.18}$$

However, a second integrable case does exist which cannot be obtained from the quadratic one through some limiting procedure. It has the autonomous form

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \frac{1 - a x_n}{1 + a x_n}. \tag{4.19}$$

The degree growth of the iterates of (4.19) is again linear, 1, 2, 2, 3, 3, 4, 4, 5, 5, ..., an indication that this mapping should be linearizable. This turns out to be the case since (4.19) is equivalent to the Gambier mapping

$$y_{n+1} + y_n = 0 \tag{4.20a}$$

$$x_n = \frac{a + y_n + x_{n-1}}{1 + a x_{n-1}}. \tag{4.20b}$$

The deautonomization of (4.19) is straightforward. We find

$$\frac{x_{n+1}x_n - 1}{x_n x_{n-1} - 1} = \frac{1 - a_n x_n}{1 + a_{n+1} x_n}, \tag{4.21}$$

where a_n is a free function of the independent variable. The associated Gambier mapping is exactly (4.20) where a is now the function a_n and not simply a constant.

5. Conclusion

In this paper, we examined integrable second-order mappings which are not of QRT type. First we have analysed mappings obtained in some previous publication [8] of two of the present authors (in collaboration with Tsuda and Takenawa) and shown that they could be extended to non-autonomous forms. The most interesting result was the deautonomization of (3.4) which was shown to be a disguised version of the q-Painlevé VI equation of Jimbo and Sakai [15]. Moreover, we extended the results of [8] by obtaining more cases of integrable non-QRT systems. All these cases were shown to be linearizable and we provided their explicit linearization in both the autonomous and non-autonomous case.

All the integrable mappings derived here as well as in [8] are of the form

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_4(x_n) - x_{n-1} f_3(x_n)}, \tag{5.1}$$

where the f_i are polynomial. The QRT case corresponds to $f_4 = f_2$ with specific forms for the f_i. It would be interesting to classify all the integrable cases of (5.1) with the help of integrability criteria. However, a brute force approach leads directly to prohibitively lengthy calculations and thus analyses such as the one presented in this paper are useful in the sense that they pave the way towards the classification of all integrable second-order mappings.

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